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Nowhere-zero flows in Cartesian bundles of graphs

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ABSTRACT

We extend the results of Imrich and Škrekovski [J. Graph Theory 43 (2003) 93–98] concerning nowhere-zero flows in Cartesian product graphs to 'twisted' Cartesian products, that is, Cartesian bundles. Our main result states that every Cartesian bundle of two graphs without isolated vertices has a nowhere-zero 4-flow.

1. Introduction

The present paper continues and extends the study of nowhere-zero flows on product graphs, initiated by Imrich and Škrekovski [4] and further advanced by Shu and Zhang [10]. Its aim is to examine a natural (although lesser known) generalisation of the Cartesian product called the Cartesian bundle. The concept was introduced in 1982 by Pisanski and Vrabec [9] and subsequently studied by several authors; see for example [3,6–8]. Given two graphs, a *base graph* B and a *fibre* F, a Cartesian bundle $B \square^{\phi} F$ with a 'twist' ϕ is a graph with vertex-set $V(B) \times V(F)$ constructed similarly to the Cartesian product $B \square F$, except that the usual adjacency $(u, x) \sim (v, x)$ between the F-layers $u \square F$ and $v \square F$ in $uv \square F$ is replaced by an adjacency $(u, x) \sim (v, \phi_{uv}(x))$ 'twisted' by some automorphism ϕ_{uv} of F. The concept of a Cartesian bundle thus embodies the idea of a graph that locally resembles the Cartesian product but globally may have a different structure.

Product graphs have been examined for many different graph properties because of their relatively simple layout and considerable generality. The first significant result concerning flows on Cartesian products of graphs is due to Imrich and Škrekovski [4]. It states that the Cartesian product of any two nontrivial connected graphs has a nowhere-zero 4-flow, and that it has a nowhere-zero 3-flow provided that both factors are bipartite. Shu and Zhang [10] later improved their result by showing that a nontrivial Cartesian product graph $G \square H$ has a nowhere-zero 3-flow except when G has a bridge and H is an odd-circuit tree. Our aim in this paper is to generalise the result of Imrich and Škrekovski to Cartesian graph bundles:

Theorem. *Every Cartesian bundle of two graphs without isolated vertices has a nowhere-zero 4-flow.*

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2. Preliminaries

We consider graphs that are finite but not necessarily simple. For a graph G we let $V(G)$ and $E(G)$ denote its vertex-set and its edge-set, respectively. Let $D(G)$ denote the set which is obtained by replacing each edge with a pair of oppositely directed *darts*; we call $D(G)$ the *dart-set* of G. If uv denotes a dart, then u is the *initial vertex* and v is the *terminal vertex* of uv. If uv is used to denote the underlying edge, the order of end-vertices is irrelevant.

Each dart z, including those on loops, has its *inverse* dart $z^{-1} \neq z$ which is incident with the same vertices but has opposite direction. For an arbitrary vertex v, we let $D(v)$ be the set of all darts emanating from v. Clearly, these sets partition the whole dart-set.

The *Cartesian product* $G \,\square\, H$ of two graphs G and H is the graph having $V(G) \times V(H)$ as its vertex-set, two vertices (v, v') and (w, w') being adjacent whenever either $v = w$ and $v'w' \in E(H)$, or $vw \in E(G)$ and $w' = v'$. In the case of multiple adjacencies or self-adjacencies the definition has to be slightly modified. For example, one can take $E(G \,\square\, H) = (V(G) \times E(H)) \cup (E(G) \times V(H))$ and define incidences in a straightforward manner. The details are left to the reader.

A *Cartesian bundle* $G \,\square^{\phi} H$ of G and H also has $V(G) \times V(H)$ as its vertex-set, but the edge-set depends on a mapping $\phi : D(G) \to \mathrm{Aut}(H)$ which to each dart z of G assigns an automorphism ϕ_z of H in such a way that $\phi_{z^{-1}} = (\phi_z)^{-1}$ for every $z \in D(G)$. In other words, ϕ is a *voltage assignment* on G with values in $\mathrm{Aut}(H)$ (see [2, Chapter 2]). Given such a mapping ϕ, two vertices (v, v') and (w, w') of $G \,\square^{\phi} H$ are defined to be adjacent whenever either $v = w$ and $v'w' \in E(H)$, or $vw \in E(G)$ and $w' = \phi_{vw}(v')$. An edge of the form $(v, v')(v, w')$ where $v \in V(G)$ and $v'w' \in E(H)$ will be referred to as an *H-edge* of $G \,\square^{\phi} H$. Similarly, an edge $(v, v')(w, w') \in E(G \,\square^{\phi} H)$ with $vw \in E(G)$ and $w' = \phi_{vw}(v')$ will be referred to as a *G-edge*, or more specifically, as a *copy* of the edge vw. Again, the definition of a Cartesian bundle readily extends to graphs with multiple edges or loops.

If the voltage assignment $\phi : D(G) \to \mathrm{Aut}(H)$ in the definition of a Cartesian bundle is the constant mapping $z \mapsto \phi_z = \mathrm{id}_H$, the resulting bundle is clearly just the usual Cartesian product; we say that such a bundle is *trivial*. In contrast to this case, a general bundle $G \,\square^{\phi} H$ need not be uniquely determined by the factors G and H, and the roles of the base and the fibre may not be interchangeable. Furthermore, $G \,\square^{\phi} H$ can be connected even when H is disconnected, in which case, however, H must have isomorphic components.

The following result can be found in [9] or [8, p. 217].

Proposition 2.1. *Let T be a tree and F be an arbitrary graph. Then every Cartesian bundle $T \,\square^{\phi} F$ is trivial; that is, $T \,\square^{\phi} F \cong T \,\square\, F$.*

We finish this section with a brief discussion of nowhere-zero flows on graphs. A *flow* on a graph G is a function $\xi : D(G) \to A$ to an abelian group A satisfying the following two conditions: (i) $\xi(z^{-1}) = -\xi(z)$ for each dart $z \in D(G)$, and (ii) $\sum_{z \in D(v)} \xi(z) = 0$ for each vertex $v \in V(G)$. A flow ξ is said to be *nowhere-zero* if $\xi(z) \neq 0$ for each dart $z \in D(G)$. A *nowhere-zero k-flow* is an integer flow which takes values from the set $\{\pm 1, \dots \pm (k-1)\} \subseteq \mathbb{Z}$. Clearly, a graph which has a nowhere-zero k-flow also has a nowhere-zero $(k+1)$-flow.

It is often convenient to describe a nowhere-zero flow on a graph as a sum of flows on subgraphs. In doing that, we will automatically view each flow on a subgraph as a flow defined on the whole graph but with zero values outside the subgraph.

For further information concerning nowhere-zero flows the reader is referred to Diestel [1, Chapter 6], Jaeger [5], or Zhang [11].

3. Proof of the main result

The basic tool which we use in the proof of our main result is vertex-splitting. Given a graph G, a vertex v of valency at least 3, and two edges $e = uv$ and $f = vw$ incident with v, we form a graph $G_{[v;e,f]}$ by deleting e and f from G and adding a new edge g joining u to w. In this case we say that $G_{[v;e,f]}$ arises from G by *splitting off* the edges e and f at v. One of the many useful properties of this operation is that a nowhere-zero k-flow in $G_{[v;e,f]}$ induces one in G. Indeed, subdividing the edge g and identifying the new vertex with v will immediately turn the original flow into a flow in G.

The next lemma is easy but helpful.

Lemma 3.1. *Let G be a nontrivial connected graph. Then there exists a series of vertex-splittings which converts G into*

(i) *one circuit, if G is eulerian, or into*
(ii) *a disjoint union of m nontrivial paths, if G has $2m$ vertices of odd valency.*

Proof. The case when G has at most two vertices of odd valency is obvious. The rest follows by induction on the number of vertices of odd valency. □

We now investigate the effect of vertex-splitting on the structure of a Cartesian bundle. Consider a bundle $G \,\square^{\phi} H$ and form the graph $G' = G_{[v;e;f]}$ by splitting off edges $e = uv$ and $f = vw$ at a vertex v of G. It is easy to see that each copy (v, t) in $G \,\square^{\phi} H$ of the vertex v of G is incident with exactly one copy e_t of e and exactly one copy f_t of f. Therefore splitting off the edges e_t and f_t at (v, t) for each vertex t of H produces a graph which itself is a Cartesian bundle with base graph G'. To be more precise, it is the Cartesian bundle $G' \,\square^{\phi'} H$ where

$$\phi'_z = \begin{cases} \phi_z & \text{for } z \in D(G') \cap D(G), \\ \phi_{vw}\phi_{uv} & \text{for } z = uw. \end{cases}$$

Since ϕ' is uniquely determined by ϕ and G', we can write $G' \,\square^{\phi} H$ instead of $G' \,\square^{\phi'} H$ without causing any confusion. Using this notational advantage we conclude that for every graph G'' that arises from a graph G by a series of vertex-splittings and for every bundle $G \,\square^{\phi} H$ there exists a unique *corresponding bundle* $G'' \,\square^{\phi} H$.

As indicated above, the corresponding bundle $G'' \,\square^{\phi} H$ arises from $G \,\square^{\phi} H$ by a series of vertex-splittings. Hence the following lemma is true.

Lemma 3.2. *Let $G \,\square^{\phi} H$ be a Cartesian bundle of two graphs G and H, and let G'' be a graph obtained from G by a series of vertex-splittings. If the corresponding bundle $G'' \,\square^{\phi} H$ has a nowhere-zero k-flow, then so does $G \,\square^{\phi} H$.*

Now we are in position to prove our main result. We proceed in two steps, establishing the existence of nowhere-zero 4-flows in trivial bundles first. Although this part has already been done by Imrich and Škrekovski [4], we give a much shorter proof.

Theorem 3.3. *The Cartesian product of two graphs without isolated vertices has a nowhere-zero 4-flow.*

Proof. Let G and H be the graphs in question. To prove the result it suffices to find a nowhere-zero 4-flow on each component of $G \,\square\, H$; we may therefore assume both G and H to be connected. By Lemma 3.1, there exists a series of vertex-splittings transforming G into a graph G' which consists of either a single circuit or a collection of disjoint nontrivial paths. Viewing $G \,\square\, H$ as a trivial bundle with base G we see that the corresponding bundle with base G' is also trivial, that is, it is the Cartesian product $G' \,\square\, H$. By Lemma 3.2 it is enough to verify that $G' \,\square\, H$ has a nowhere-zero 4-flow. To this end, we observe that $G' \,\square\, H \cong H \,\square\, G'$, so we can apply a similar splitting procedure to H within the direct product $H \,\square\, G'$. As a result, we obtain a Cartesian product $H' \,\square\, G'$ where each connected component is either a Cartesian product of two circuits, a Cartesian product of two paths, or a Cartesian product of a path and a circuit. It is an easy exercise to prove that the product of two paths has a nowhere-zero 3-flow whereas the product of a path and a circuit or the product of two circuits has a nowhere-zero 4-flow. From Lemma 3.2 we finally obtain a nowhere-zero 4-flow on $G' \,\square\, H$ and consequently also on $G \,\square\, H$. □

We now proceed to the general case. It is somewhat unfortunate that the idea of constructing a nowhere-zero 4-flow by applying the operation of vertex-splitting in both the base and the fibre does not extend beyond trivial bundles. The difficulty here is that splitting off a pair of adjacent edges of the fibre may significantly alter its automorphism group. As a consequence, the voltage assignment $\phi: D(G) \to \mathrm{Aut}(H)$ needed to define a bundle $G \,\square^{\phi} H$ may be completely destroyed.

Theorem 3.4. *Every Cartesian bundle of two graphs without isolated vertices has a nowhere-zero 4-flow.*

Proof. Let $G \,\square\,^{\phi}H$ be a Cartesian bundle where both G and H have minimum valency at least 1. Without loss of generality we may assume that $G \,\square\,^{\phi}H$ is connected. This implies that G is connected and that H has no isolated vertices (although it may happen to be disconnected).

If each component of both G and H is Eulerian, then so are the components of the bundle, because the valency of each vertex (u, v) in $G \,\square\,^{\phi}H$ is the sum of valencies of $u \in V(G)$ and $v \in V(H)$. Hence $G \,\square\,^{\phi}H$ has a nowhere-zero 2-flow.

If G is not Eulerian (and H is arbitrary), we apply Lemma 3.1 to split G into a graph G' consisting of several disjoint paths. Proposition 2.1 shows that $G' \,\square\,^{\phi}H$ is isomorphic to the Cartesian product $G' \,\square\, H$, which, by Theorem 3.3, has a nowhere-zero 4-flow. Lemma 3.2 then implies that $G \,\square\,^{\phi}H$ has a nowhere-zero 4-flow, too.

Thus we are left with the case where G is Eulerian and H is not. In order to construct a nowhere-zero 4-flow on $G \,\square\,^{\phi}H$ we express $G \,\square\,^{\phi}H$ as a union of two subgraphs K and L and get the required flow as a combination of a flow on K and a flow on L.

Let K be the subgraph obtained by removing all H-edges from $G \,\square\,^{\phi}H$, and let L be the subgraph $(G - e) \,\square\,^{\phi}H$ where e is an arbitrary fixed edge of G. Clearly, each edge of $G \,\square\,^{\phi}H$ is contained in at least one of K and L and each G-edge, except for the copies of e, is in both. The key step is to choose a nowhere-zero flow σ on K and a nowhere-zero flow τ on L in such a way that $\sigma(z) \neq -\tau(z)$ for each dart z which belongs to both K and L. This will guarantee that $\sigma + \tau$ is a nowhere-zero flow.

Let us examine the subgraph K first. Each component of K is an Eulerian graph, so K has a nowhere-zero 2-flow, say π. To find an appropriate flow for K consider an arbitrary edge $(v, v')(w, w')$ in K. Clearly, $(v, v')(w, w')$ is a G-edge in $G \,\square\,^{\phi}H$ where $w' = \phi_{vw}(v')$ for some automorphism ϕ_{vw} of H. Hence v' and w' have the same valency while v and w have even valency, because G is Eulerian. It follows that in $G \,\square\,^{\phi}H$ the end-vertices of each edge of K have valencies of the same parity, so we may classify the edges of K into *even edges* or *odd edges* depending on whether their end-vertices have even or odd valency in $G \,\square\,^{\phi}H$, respectively. Since each component of K must have edges of the same parity, the components can be classified as *even* or *odd* accordingly. We are now ready to define a flow σ on K as follows: set $\sigma(z) = \pi(z)$ whenever z is a dart from an even component of K and $\sigma(z) = 2\pi(z)$ if z is a dart from an odd component of K.

As regards the subgraph $L = (G - e) \,\square\,^{\phi}H$, Lemma 3.1 guarantees that there is a series of vertex-splittings that converts $G - e$ into a single path, say P. The corresponding bundle $P \,\square\,^{\phi}H$ is isomorphic to the Cartesian product $H \,\square\, P$, so we can proceed with splittings in H to obtain a Cartesian product $H' \,\square\, P$ where each component is the product of two nontrivial paths. It follows that $H' \,\square\, P$ has a nowhere-zero 3-flow, and by Lemma 3.2 so does the subgraph L.

Nevertheless, we need a more specific 3-flow on L. In order to construct one, it is convenient to think of every component of $H' \,\square\, P$ as a planar grid with inner faces being quadrilaterals. We orient the interior of each quadrilateral in such a way that adjacent quadrilaterals have opposite orientation and, for each quadrilateral q, we send the value 1 around the boundary of q in the direction of its orientation. This produces a nowhere-zero 2-flow τ_q on the bounding 4-cycle of any given quadrilateral q. It is easy to see that the sum $\tau' = \sum_q \tau_q$ of all these flows becomes a nowhere-zero 3-flow on $H' \,\square\, P$. In turn, τ' induces a nowhere-zero 3-flow on L which we denote by τ.

It remains to show that $\sigma + \tau$ is a nowhere-zero 4-flow on $G \,\square\,^{\phi}H$. From the description of the flow τ' it is clear that the darts on the outer edges of each grid carry values ± 1 whereas those on the inner edges carry values ± 2. Since vertex-splittings preserve the parity of the valency of an arbitrary vertex, every outer P-edge of $H' \,\square\, P$ corresponds to an odd G-edge of $G \,\square\,^{\phi}H$ while every inner P-edge of $H' \,\square\, P$ corresponds to an even G-edge of $G \,\square\,^{\phi}H$. Therefore, a dart z on a G-edge of L has $\tau(z) = \pm 2$ provided that the edge is even, and has $\tau(z) = \pm 1$ if the edge is odd. Summing up, for every dart z from $K \cap L$ we have $|\sigma(z)| \neq |\tau(z)|$ and consequently $\sigma(z) \neq -\tau(z)$. It follows that $\sigma + \tau$ is a nowhere-zero flow on $G \,\square\,^{\phi}H$ with $|\sigma(z) + \tau(z)| \leq 1 + 2$, and therefore a nowhere-zero 4-flow. The proof is now complete. $\square$

4. Final remarks

The fundamental property that lends the concept of a Cartesian bundle its generality while offering sufficient control over the structure consists in the fact that the perfect matching between adjacent

H-layers $u \square H$ and $v \square H$ in $G \square^{\phi} H$ is determined by an automorphism ϕ_{uv} of H. If this property is abandoned and just any perfect matching is permitted, the conclusion of the main result may fail.

As an example consider a Cartesian bundle $K_2 \square^{\phi} C_5$ with base the complete graph on two vertices and with fibre the 5-circuit. By Theorem 3.4, the bundle admits a nowhere-zero 4-flow. Nevertheless, if the perfect matching between the two 5-circuits in $K_2 \square^{\phi} C_5$ determined by ϕ is replaced by a perfect matching that creates the Petersen graph, a nowhere-zero 4-flow can no longer be guaranteed.

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